Integrals of Exponential and Logarithmic Functions
Integration Guidelines

1. Learn your rules (Power rule, trig rules, log rules, etc.).
2. Find an integration formula that resembles the integral you are trying to solve (u-substitution should accomplish this goal).
3. If \( u \)-substitution does not work, you may need to alter the integrand (long division, factor, multiply by the conjugate, separate the fraction, or other algebraic techniques).
4. When all else fails, use your TI-89.
Power Rule for Integrals

\[ \int x^n \, dx = \frac{1}{n+1} x^{n+1} + C \]

for \( n \neq -1 \)

The power rule is not valid for \( n = -1 \) because:

\[ \int x^{-1} \, dx = \frac{1}{-1+1} x^{-1+1} + C = \frac{x^0}{0} + C \]

(meaningless)
Derivative and Antiderivatives that Deal with the Natural Log

However, we know the following to be true:

\[ \frac{d}{dx} \ln x = \frac{1}{x} \]

This shows that \( \ln x \) is an antiderivative of \( \frac{1}{x} \).

Or:

\[ \int x^{-1} \, dx = \int \frac{dx}{x} = \ln x + C \]

But, thus formula is only valid for \( x > 0 \) (where \( \ln x \) is defined). How can we have an antiderivative on its full domain?
Indefinite Integral of $y = 1/x$

If $x \neq 0$, then

$$\int \frac{dx}{x} = \ln |x| + C$$

This is valid because we proved the following result:

$$\frac{d}{dx} \ln |g(x)| = \frac{g'(x)}{g(x)}$$
Derivative and Antiderivatives that Deal with the Exponentials

We know the following to be true:

\[
\frac{d}{dx} a^x = (\ln a) \cdot a^x
\]

Solve for \(a^x\):

\[
\frac{1}{\ln a} \frac{d}{dx} a^x = a^x
\]

\[
\frac{d}{dx} \left( \frac{1}{\ln a} \cdot a^x \right) = a^x
\]

(Constant Rule in reverse)

This shows the antiderivative of \(a^x\):

\[
\int a^x \, dx = \frac{1}{\ln a} \cdot a^x
\]

As long as \(a > 0\) (where \(\ln a\) is defined), this antiderivative satisfies all values of \(x\).
If $a > 0$, then

$$\int a^x \, dx = \frac{1}{\ln a} \cdot a^x$$

If $a = e$, then

$$\int e^x \, dx = e^x$$
Example 1

Find \( \int xe^{3x^2-1} \, dx \).

Define \( u \) and \( du \):

\[
u = 3x^2 - 1 \quad \Rightarrow \quad du = 6x \cdot dx
\]

Substitute to replace EVERY \( x \) and \( dx \):

\[
\int xe^{3x^2-1} \, dx = \int xe^u \, dx = \int x e^u \frac{1}{6x} \, du
\]

\[
= \frac{1}{6} \int e^u \, du
\]

\[
= \frac{1}{6} e^u + C
\]

Substitute back to leave your answer in terms of \( x \).

\[
= \frac{1}{6} e^{3x^2-1} + C
\]
Example 2

Find \( \int \frac{dx}{x \ln x} \) for \( x > 1 \).

Define \( u \) and \( du \):

\[ u = \ln x \]
\[ du = \frac{1}{x} \cdot dx \]

Substitute to replace EVERY \( x \) and \( dx \):

\[ \int \frac{dx}{x \ln x} = \int \frac{dx}{x \cdot u} \]
\[ = \int \frac{x \cdot du}{x \cdot u} \]
\[ = \int \frac{du}{u} \]
\[ = \ln |u| + C \]

Substitute back to Leave your answer in terms of \( x \).

\[ = \ln |\ln x| + C = \ln (\ln x) + C \]
Example 3

Find \( \int \frac{x+3}{x^2 + 5x + 6} \, dx \)

Do not forget about all of your old techniques

\[
\int \frac{x+3}{x^2 + 5x + 6} \, dx = \int \frac{x+3}{(x+3)(x+2)} \, dx = \int \frac{1}{x+2} \, dx
\]

\( u = x + 2 \quad du = dx \)

\[
= \int \frac{1}{u} \, du = \ln |u| + C = \ln |x + 2| + C
\]
Example 4

Find \( \int \frac{x^2-x-2}{x-3} \, dx \)

<table>
<thead>
<tr>
<th>Perform Polynomial Division.</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x )</td>
</tr>
<tr>
<td>( x )</td>
</tr>
<tr>
<td>( x^2 )</td>
</tr>
</tbody>
</table>

Thus:

\[
\int \frac{x^2-x-2}{x-3} \, dx = \int (x + 2) \, dx + \int \frac{4}{x-3} \, dx
\]

\[
= \frac{1}{2} x^2 + 2x + C + 4 \int \frac{1}{x-3} \, dx
\]

\[
= \frac{1}{2} x^2 + 2x + 4 \ln |x-3| + C
\]